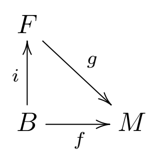
Friday November 8 Lecture Notes

1 Projective Modules

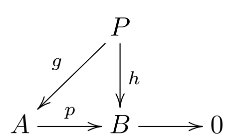
A universal property characterization of free modules:

Proposition Let F be a free R modules and let B be a base for F. Suppose $i : B \to F$ is the inclusion map. Let M be any R module and let $f : B \to M$ be any function, then there exists unique g (R module homomorphism) s.t.



commutes.

Definition An R module P is projective if for all surjective R module homomorphisms $p : A \to B$ and all R module homomorphisms $h : P \to B$, there exists a unique g s.t.



commutes.

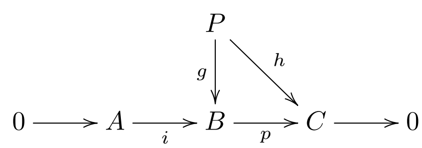
Proposition Free modules are projective.

Proof Let F be a free module with base $B = \{b_i : i \in I\}$. Suppose we have R module homomorphisms $p : A \to C$ (surjective) and $h : F \to C$. Since p is surjective, for each $i \in I$, there is $a_i \in A$ s.t. $p(a_i) = h(b_i)$. By freeness, there exists $g : F \to A$ with $b_i \mapsto a_i$.

Definition A functor $T : R\text{-mod} \to \text{Grp}_{\text{Abel}}$ is exact if whenever $0 \to A \xrightarrow{i} B \xrightarrow{p} C \to 0$ is exact, then $0 \to T(A) \xrightarrow{T_i} T(B) \xrightarrow{T_p} T(C) \to 0$ is exact.

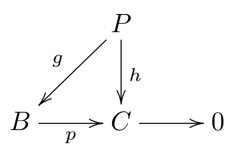
Proposition An R module P is projective iff $\text{Hom}_R(P, \_)$ is exact.

Proof Take an exact sequence $0 \to A \xrightarrow{i} B \xrightarrow{p} C \to 0$ (regardless of the assumption on P, $0 = \mathrm{Hom}_R(P, 0) \to \mathrm{Hom}_R(P, A) \xrightarrow{i_*} \mathrm{Hom}_R(P, B) \xrightarrow{p_*} \mathrm{Hom}_R(P, C)$ is exact). Note that $\ker i_* = \{f \in \mathrm{Hom}_R(P, A) : if = 0\} = \{f \in \mathrm{Hom}_R(P, A) : f(P) \subseteq \ker i\} = 0$. Now, $\ker p_* = \{g \in \mathrm{Hom}_R(P, B) : pg = 0\} = \{g \in \mathrm{Hom}_R(P, B) : g(P) \subseteq \ker p\} = \{g \in \mathrm{Hom}_R(P, B) : g(P) \subseteq \mathrm{im}\, i\}$, but i is injective, so for all $q \in P$, there exists a unique $a_q \in A$ s.t. $g(q) = i(a_q)$, so define $h \in \mathrm{Hom}_R(P, A)$ by $h(q) = a_p$. We can check that this is a module map homomorphism, and that $ih = g$, so $\ker p_* = \mathrm{im}\, i_*$. Suppose P is projective. Given $h : P \to C$, by projective there is g s.t.



commutes. So $h = pg = p_* g$, and thus p_* is onto, so $\mathrm{Hom}_R(P, \_)$ is exact.

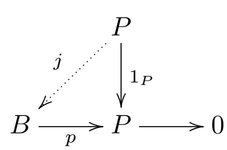
Suppose $\mathrm{Hom}_R(P, \_)$ is exact. if $p : B \to C$ is a surjection, then $0 \to \ker p \to B \xrightarrow{p} C \to 0$ is exact. So P_* is surjective, and given $h : P \to C$, there is g with $p_* g = h$, i.e.,



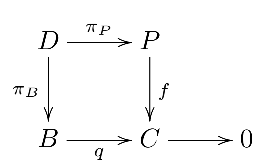
commutes. So p is projective.

Proposition Let P be an R module. Then P be projective iff every short exact sequence $0 \to A \xrightarrow{i} B \xrightarrow{p} C \to 0$ splits.

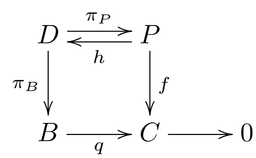
Proof Assume P is projective. Then there exists j s.t.



commutes, but $pj = 1_P$, so j is the splitting map. Now suppose every short exact sequence of the given form splits. Take a surjection $q : B \to C$ and $f : P \to C$, and consider



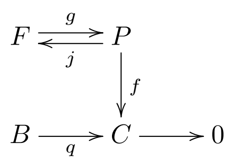
where $D = \{(b,p) : b \in B, p \in P, q(b) = f(p)\}$ is an R module and it has two coordinate projections π_B and π_P (as it turns out, this is a categorical pullback). Surjectivity of q implies that for any $p \in P$, there is b s.t. $q(b) = f(p)$, so π_P is surjective, and $0 \to \ker \pi_P \to D \xrightarrow{\pi_P} P \to 0$ splits by hypothesis, by $h : P \to D$, say, s.t.



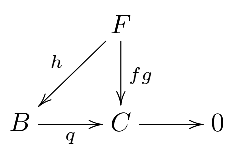
commutes. So $\pi_B h$ shows that p is projective.

Proposition An R module P is projective iff P is a direct summand of a free module.

Proof Suppose P is projective. Every module is a quotient of a free module, so, in particular, there is some surjection $g : F \to P$ where F is free. So $0 \to \ker g \to F \to P \to 0$ is exact, so it splits, and $F \simeq P \oplus \ker g$. Assume P is a direct summand of a free module F. Then, we have a surjection $g : F \to P$ and $j : P \to F$, and $gj = 1_P$. Suppose there is a surjection $q : B \to C$ and a map $f : P \to C$ s.t.



commutes. Since F is free, it is projective, and so there is h s.t.



commutes, and thus hj satisfies the definition of projectivity for P.

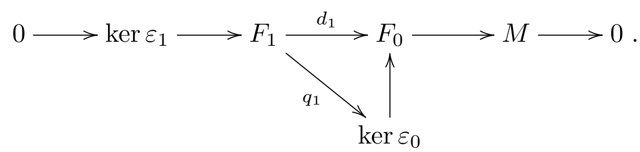
e.g. Take $R = \mathbb{Z}/6\mathbb{Z} = \{0,1,2,3,4,5\}$, and let $I = \{0,3\}$ and $J = \{0,2,4\}$ be ideals of R. Then $R = I \oplus J$, but neither I, nor J are free as R modules, but they are projective.

2 Chain Complexes

Definition Let M be an R module. A projective resolution of M is an exact sequence $\cdots \to P_2 \to P_1 \to P_0 \to M \to 0$, where $P_0, P_1, \ldots$ are projective modules. A projective resolution is a free resolution if $P_0, P_1, \ldots$ are free.

Proposition Every module M has a free resolution.

Proof Choose a spanning set of M, and let F_0 be the free module on this set. Consider $0 \to \ker \varepsilon_0 \to F_0 \xrightarrow{\varepsilon_0} M \to 0$, where ε_0 takes the spanning set to itself. Also, $0 \to \ker \varepsilon_1 \to F_1 \xrightarrow{\varepsilon_1} \ker \varepsilon_0 \to 0$, and



Then $\operatorname{im} d_1 = \operatorname{im} \varepsilon_1 = \ker \varepsilon_0$, which proves exactness. Moreover, $\ker d_1 = \ker \varepsilon_1$, which also proves exactness. Continue this process to obtain a free resolution.

Definition A chain complex $C_\bullet$ (or $(C_\bullet, d_\bullet)$) is a sequence of modules and maps for $n \in \mathbb{Z}$, $\cdots \to C_{n+1} \xrightarrow{d_{n+1}} C_n \xrightarrow{d_n} C_{n-1} \xrightarrow{d_{n-1}} \cdots$, in which $d_n d_{n+1} = 0$. Here, d_n is called differentiation.

Note: $d_n d_{n+1} = 0$ iff $\operatorname{im} d_{n+1} \subseteq \ker d_n$

e.g. An exact sequence, infinite in both directions, is a chain complex.

e.g. $\cdots \to 0 \to 0 \to 0 \to \cdots$ (the zero complex $0_\bullet$)

e.g. An exact sequence which ends with 0 can be extended: $\cdots \to 0 \to 0 \to 0 \to \cdots$

e.g. Let $C_\bullet = \cdots \to C_{n+1} \xrightarrow{d_{n+1}} C_n \xrightarrow{d_n} C_{n-1} \xrightarrow{d_{n-1}} \cdots$ be a chain complex, and let $F : R\text{-mod} \to R\text{-mod}$ be a functor with the additive property: $F(f+g) = F(f) + F(g)$ for morphism f and g. Then

$$F(C_\bullet) = \cdots \to F(C_{n+1}) \xrightarrow{F(d_{n+1})} F(C_n) \xrightarrow{F(d_n)} F(C_{n-1}) \xrightarrow{F(d_{n-1})} \cdots$$

is a chain complex because the additive property implies that $F(f+0) = F(f) + F(0) = F(f)$, i.e., $F(0) = 0 = F(d_n d_{n+1}) = F(d_n)F(d_{n+1})$ (since F is a functor).

Note: $F(C_\bullet)$ may not be exact even if $C_\bullet$ is exact.

Definition If $(C_\bullet, d_\bullet)$ is a chain complex, then n-cycles, $Z_n(C_\bullet) = \ker d_n \subseteq C_n$, and n-boundaries $B_n(C_\bullet) = \operatorname{im} d_{n+1} \subseteq C_n$.

Note: $\operatorname{im} d_{n+1} \subseteq \ker d_n$, so $B_n(C_\bullet) \subseteq Z_n(C_\bullet)$.

Definition If $C_\bullet$ is a chain complex, then the nth homology $H_n(C_\bullet) = Z_n(C_\bullet)/B_n(C_\bullet)$.

Note: We can make a category of chain complexes. The objects will be chain complexes of R modules, and the morphism will be chain maps. For each $n \in \mathbb{Z}$, H_n is a functor from chain complexes to R modules.

We saw what H_n does on objects, but what about the morphisms?

Given

$$\begin{array}{ccccccccc} \cdots & \longrightarrow & C_{n+1} & \xrightarrow{d_{n+1}} & C_n & \xrightarrow{d_n} & C_{n-1} & \longrightarrow & \cdots \\ & & \downarrow{\scriptstyle f_{n+1}} & & \downarrow{\scriptstyle f_n} & & \downarrow{\scriptstyle f_{n-1}} & & \\ \cdots & \longrightarrow & C'_{n+1} & \xrightarrow{d'_{n+1}} & C'_n & \xrightarrow{d'_n} & C'_{n-1} & \longrightarrow & \cdots \end{array}$$

we define $H_n(f_\bullet) : H_n(C_\bullet) \to H_n(C'_\bullet)$ iff $H_n(f_\bullet) = Z_n(C_\bullet)/B_n(C_\bullet) \to Z_n(C'_\bullet)/B_n(C'_\bullet)$ by $z_n + B_n(C_\bullet) \mapsto f_n(z_n) + B_n(C'_\bullet)$.

Version: 1.1